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VERTEX EXPONENTS OF TWO-COLORED PRIMITIVE EXTREMAL MINISTRONG DIGRAPHS

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Abstract
A two-colored digraph $D^{(2)}$ is a digraph D whose each of its arcs is colored by either red or blue. A two-colored digraph $D^{(2)}$ is primitive provided that there is a positive integer $h+k$ such that any pair of vertices in $D^{(2)}$ can be connected by a walk of length $h+k$ consisting of h red arcs and k blue arcs. The smallest of such positive integer $h+k$ is the exponent of $D^{(2)}$ and is denoted by $\exp(D^{(2)})$. The exponent of a vertex v in a two-colored digraph $D^{(2)}$ is the smallest positive integer $s+t$ such that for each vertex x in $D^{(2)}$ there is a walk of length $s+t$ consisting of s red arcs and t blue arcs. In this paper we discuss the vertex exponents of a primitive two-colored extremal ministrong digraph $D^{(2)}$ on n vertices. If $D^{(2)}$ has one blue arc, we show that the exponents of vertices of $D^{(2)}$ lie on $[n^2 - 5n + 8, n^2 - 3n + 1]$. If $D^{(2)}$ has two blue arcs, we show that the exponents of vertices in $D^{(2)}$ lie on $[n^2 - 4n + 4, n^2 - n]$.

Keywords: extremal ministrong digraph, two-colored digraphs, primitive digraphs, exponents, vertex exponents

1. Introduction

A digraph D is *strongly connected* provided that for each pair of vertices u and v in D there is a walk from u to v and a walk from v to u. A strongly connected digraph D is said to be *ministrong* if each digraph obtained from D by mean of removal any arc of D will result in a not strongly connected digraph. A strongly connected digraph D is *primitive* provided there exists a positive integer ℓ such that for every pair of not necessarily distinct vertices u and v in D there is a walk from u to v of length ℓ. The smallest of such positive integer ℓ is the *exponent* of D and is denoted by $\exp(D)$. Exponents of primitive digraphs have been studied extensively because of their importance not only in graph theory but also in matrix theory and their application in communications [23]. Results on exponents of digraph can be found in [3]. By an extremal ministrong digraph on n vertices we mean a primitive ministrong digraph with exponent equals $n^2 - 4n + 6$.

Brualdi and Liu [1] generalized the concept of exponent of a primitive digraph by defining more *local exponents* as follows. Let D be a primitive digraph, the exponent of a vertex v in D is the smallest positive integer t such there is a walk of length t from the vertex v to all vertices in D. The exponent of a vertex v is denoted by $\gamma_D(v)$. Let the vertices $v_1, v_2, \ldots, v_n$ of the digraph D be ordered such that we have $\gamma_D(v_1) \le \gamma_D(v_2) \le \ldots \le \gamma_D(v_n)$. For $1 \le k \le n$, the number $\gamma_D(v_k)$ is called the first k^{th} *generalized exponent* of D and is denoted by $\gamma_D(k)$. The readers interested in the vertex exponents of primitive digraphs should consult the literatures (see [4,10,12,13,15]). We mention here that the number $\gamma_D(k)$ has a nice interpretation in the model a memory less communication networks (see [4]).

By a *two-colored* digraph $D^{(2)}$ (a 2-digraph for short) we mean a digraph D such that each of its arcs is colored by either red or blue but not both colors. Let s and t be nonnegative integers. By an (s,t)-walk we mean a walk of length $s + t$ consisting of s red arcs and t blue arcs. For a walk w in $D^{(2)}$ we respectively define $r(w)$ and $b(w)$ to be the number of red and blue arcs contained in w. The vector $\begin{bmatrix} r(w) \\ b(w) \end{bmatrix}$ is called the composition of the walk w and $\ell(w) = r(w) + b(w)$ is the length of the walk w. A 2-digraph $D^{(2)}$ is primitive provided there are nonnegative integers h and k such that for each pair of vertices u and v in $D^{(2)}$ there is an (h,k)-walk from u to v. The smallest positive integer $h + k$ over all such nonnegative integers h and k is the *exponent* of $D^{(2)}$ and denoted by $\exp(D^{(2)})$. The study of exponents of two-colored digraph is initiated by Shader and Suwilo [14]. Since then many researches on exponents of two-colored digraphs have been conducted (see [6, 8, 9, 16-18]).

Let $D^{(2)}$ be a strongly connected 2-digraph and let $C = \{C_1, C_2, \ldots, C_q\}$ be the set of all cycles in $D^{(2)}$. Define a *cycle matrix* of $D^{(2)}$ to be a 2 by q matrix

$$M = \begin{bmatrix} r(C_1) & r(C_2) & \cdots & r(C_q) \\ b(C_1) & b(C_2) & \cdots & b(C_q) \end{bmatrix}, \qquad (1)$$

that is M is a matrix such that its ith column is the composition of the ith cycle C_i, $i = 1, 2, \ldots, q$. If the rank of M is 1, the content of M is defined to be 0, and otherwise the content of M is the greatest common divisor of the 2 by 2 minors of M. The following result, due to Fornasini and Valcher [5], gives algebraic characterization of a primitive 2-digraph.

Theorem 1.1: [5] *Let $D^{(2)}$ be a strongly connected 2-digraph with at least one arc of each color. Suppose the cycle matrix of $D^{(2)}$ is M. The 2-digraph $D^{(2)}$ is primitive if and only if the content of M is 1.*

By an underlying digraph of a 2-digraph $D^{(2)}$ is a digraph D obtained from $D^{(2)}$ by ignoring the color of each arc in $D^{(2)}$.

Let $D^{(2)}$ be a primitive 2-digraph on n vertices $v_1, v_2, \ldots, v_n$. Gao and Shao [7] extended the definition of vertex exponent of a digraph into vertex exponent of a 2-digraph. For a vertex u in $D^{(2)}$ the vertex exponent of u is the smallest positive integer $s + t$ such that for each vertex v in $D^{(2)}$ there is a (s,t)-walk from u to v. Let the vertices $v_1, v_2, \ldots, v_n$ of $D^{(2)}$ be ordered such that

$$\gamma_{D^{(2)}}(v_1) \le \gamma_{D^{(2)}}(v_2) \le \cdots \le \gamma_{D^{(2)}}(v_n).$$

For $1 \le k \le n$, the number $\gamma_{D^{(2)}}(v_k)$ is called the first kth generalized exponent of $D^{(2)}$ and is denoted by $\gamma_{D^{(2)}}(k)$. Gao and Shao [7] give a formula for vertex exponent of primitive 2-digraph of Wielandt type on n vertices. That is a two-colored digraph whose underlying digraph is the primitive digraph consisting of the cycle $v_1 \to v_n \to v_{n-1} \to \ldots \to v_2 \to v_1$ of length n and the arc $v_1 \to v_{n-1}$. For a primitive two-colored Wielandt digraphs $W^{(2)}$ they show that: (i) $\gamma_{W^{(2)}}(v_k) = n^2 - 2n + k - j + 1$, if $W^{(2)}$ has only one blue arc of the form $v_j \to v_{j-1}$ where $2 \le j \le n-1$, (ii) $\gamma_{W^{(2)}}(v_k) = n^2 - 2n + k$, if $W^{(2)}$ has two blue arcs of the form $v_1 \to v_{n-1}$ and $v_1 \to v_n$ and (iii) $\gamma_{W^{(2)}}(v_k) = n^2 - 2n + k$, if $W^{(2)}$ has two blue arcs $v_1 \to v_{n-1}$ and $v_n \to v_{n-1}$.

This paper discusses the vertex exponents of two-colored extremal ministrong digraphs on n vertices. In Section 2 we discuss previous works on exponents of primitive extremal ministrong digraphs. In Section 3 we discuss a way to set up a lower bound and an upper bound for vertex exponent of a 2-digraphs consisting of two cycles. In Section 4 we present our main result on vertex exponent of two-colored extremal ministrong digraphs. We note that this paper is a completed version of [19].

2. Previous works on exponents and vertex exponents of ministrong digraph

In this section we discuss some results on exponents and vertex exponents of ministrong digraphs and ministrong 2-digraphs. We begin with the following result of Brualdi and Ross [2] on exponents of primitive ministrong digraphs.

Theorem 2.1: [2] *Let D be a ministrong digraph on n vertices $v_1, v_2, \ldots, v_n$. Then*

$$6 \le \exp(D) \le n^2 - 4n + 6.$$

The upper bound is achieved if and only if D is isomorphic to the digraph consisting the cycle

$$v_1 \to v_{n-2} \to v_{n-3} \to \ldots \to v_2 \to v_1$$

and the path $v_1 \to v_n \to v_{n-1} \to v_{n-3}$

Since then many researches on exponents and generalized exponents of primitive ministrong digraph have been conducted. Literatures on exponents an vertex exponents of primitive ministrong digraphs can be found for examples in [11, 20-23].

Let D be the extremal primitive ministrong digraph on n vertices and let $D^{(2)}$ be a 2-digraph obtained by coloring the arcs of D with red or blue. We note that the 2-digraph $D^{(2)}$ consists of two cycles, namely the cycle

$$C_1 : v_1 \to v_n \to v_{n-1} \to v_{n-3} \to v_{n-4} \to \ldots \to v_2 \to v_1$$

of length $n - 1$ and the cycle

$$C_2 : v_1 \to v_{n-2} \to v_{n-3} \to v_{n-4} \to \ldots \to v_2 \to v_1$$

of length $n - 2$. Lee and Yang [9] show that the two-colored extremal ministrong digraph $D^{(2)}$ is primitive if and only if the cycle matrix of $D^{(2)}$ is

$$M = \begin{bmatrix} r(C_1) & r(C_2) \\ b(C_1) & b(C_2) \end{bmatrix} = \begin{bmatrix} n-2 & n-3 \\ 1 & 1 \end{bmatrix}. \quad (2)$$

The following theorem of Lee and Yang [9] gives a bound for exponents of 2-digraphs whose underlying digraph is the extremal ministrong digraph on n vertices with exponent $n^2 - 4n + 6$.

Theorem 2.2: [9] *Let D be the primitive ministrong digraph on n vertices with exponent $n^2 - 4n + 6$. Let $D^{(2)}$ be a primitive 2-digraph whose underlying digraph is D. Then*

$$2n^2 - 8n + 7 \le \exp(D^{(2)}) \le 2n^2 - 5n + 3$$

3. Bounds for Vertex Exponents of two-colored digraphs

In this section we discuss a way to set up a lower and an upper bound for vertex exponents of primitive 2-digraphs. We start by discussing an upper bound. For the rest of the paper we assume that the exponent of the vertex v_k is obtained using (s, t)-walks.

Proposition 3.1 *Let $D^{(2)}$ be a primitive 2-digraph and let v_k be a vertex in $D^{(2)}$. If for some nonnegative integers s and t and some paths p_{ki} from v_k to v_i, $i = 1, 2, \ldots, n$ the system of equations*

$$M\mathrm{x} + \begin{bmatrix} r(p_{k,i}) \\ b(p_{k,i}) \end{bmatrix} = \begin{bmatrix} s \\ t \end{bmatrix} \quad (3)$$

has a nonnegative integer solution, then the exponent $\gamma_{D^{(2)}}(v_k) \le s + t$

Proof: Let M be a 2 by t cycle matrix of $D^{(2)}$ as in equation (1). For every vertex v_i, $i = 1, 2, \ldots, n$ in $D^{(2)}$ we claim that there is an (s, t)-walk from v_k to v_i. Let x be the solution of the system (3). Since $x = (x_1, x_2, \ldots, x_q)^T$ is a nonnegative integer vector, the walk that starts at v_k, moves to v_i along the path p_{ki} and along the way moves x_j times around the cycle C_j for $j = 1, 2, \ldots, q$ is an (s, t)-walk from v_k to v_i. By definition of vertex exponent we have that $\gamma_{D^{(2)}}(v_k) \le s + t$.

In the next proposition, we describe an upper bound of vertex exponent of any vertex in a 2-digraph $D^{(2)}$ in term of the vertex exponent of a specified vertex. In Proposition 3.2 the notion $d(v_k, v)$ means the length of the shortest path from v_k to v.

Proposition 3.2 *Let $D^{(2)}$ be a primitive 2-digraph with vertices $v_1, v_2, \ldots, v_n$, and let v be a vertex in $D^{(2)}$ with exponent $\gamma_{D^{(2)}}(v)$. Then for each vertex v_k we have $\gamma_{D^{(2)}}(v_k) \le \gamma_{D^{(2)}}(v) + d(v_k, v)$.*

Proof: Let $p_{k,v}$ be the $(r(p_{k,v}), b(p_{k,v}))$-path from v_k to v with length $d(v_k, v)$. Since the exponent of the vertex v is $\gamma_{D^{(2)}}(v)$, there is an (s, t)-walk of length $\gamma_{D^{(2)}}(v) = s + t$ from vertex v to each vertex v_i, $j = 1, 2, \ldots, n$. This implies for each vertex v_k in $D^{(2)}$ there is an $(s+r(p_{k,v}), t+b(p_{k,v}))$-walk from the vertex v_k to each vertex v_j, $j = 1, 2, \ldots, n$, namely the walk that starts at v_k, moves to v along the $(r(p_{k,v}), b(p_{k,v}))$-path and then moves to v_j by using an (s, t)-walk from v to v_j. Now, we conclude $\gamma_{D^{(2)}}(v_k) \le \gamma_{D^{(2)}}(v) + d(v_k, v)$.

The following lemma presents a way to set up a lower bound for vertex exponent of a primitive 2-digraph consisting of two cycles.

Lemma 3.3 *Let $D^{(2)}$ be a primitive 2-digraph consisting two cycles with cycle matrix* $M = \begin{bmatrix} r(C_1) & r(C_2) \\ b(C_1) & b(C_2) \end{bmatrix}$. *Let v_k be any vertex in $D^{(2)}$ and suppose there is an (s, t)-walk from v_k to each vertex v_i. in $D^{(2)}$ with* $\begin{bmatrix} s \\ t \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix}$ *for nonnegative integers u and v. Then* $\begin{bmatrix} u \\ v \end{bmatrix} \ge M^{-1} \begin{bmatrix} r(p_{ki}) \\ b(p_{ki}) \end{bmatrix}$ *for some path p_{ki} from v_k to v_i.*

Proof: Let p_{ki} be a path from v_k to v_i. Since every walk can be decomposed into cycles and a path, we have

$$\begin{bmatrix} s \\ t \end{bmatrix} = M\text{x} + \begin{bmatrix} r(p_{ki}) \\ b(p_{ki}) \end{bmatrix} \tag{4}$$

For some nonnegative integer vector x; We note that since $D^{(2)}$ is primitive, M is an invertible matrix. By considering $\begin{bmatrix} s \\ t \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix}$ and equation (4) we now have

$$x = \begin{bmatrix} u \\ v \end{bmatrix} - M^{-1} \begin{bmatrix} r(p_{ki}) \\ b(p_{ki}) \end{bmatrix} \ge 0 \tag{5}$$

Hence from (5) we have

$$\begin{bmatrix} u \\ v \end{bmatrix} \ge M^{-1} \begin{bmatrix} r(p_{ki}) \\ b(p_{ki}) \end{bmatrix}$$

and the lemma holds.

As a direct consequence of Lemma 3.3 we have the following lower bound for vertex exponents.

Corollary 3.4 *Let $D^{(2)}$ be a primitive two-colored digraph consisting of two cycles C_1 and C_2. Let v_k be a vertex in $D^{(2)}$ and let $p_{k,i}$ and $p_{k,j}$ be path from v_k to v_i and from v_k to v_i with $i \ne j$. If $u_0 = b(C_2)r(p_{k,i}) - r(C_2)b(p_{k,i}) \ge 0$ and $v_0 = r(C_1)b(p_{k,j}) - b(C_1)r(p_{k,j}) \ge 0$, then $\gamma_{D^{(2)}}(v_k) \ge \ell(C_1)u_0 + \ell(C_2)v_0$.*

Proof: We assume that the exponent of v_k can be achieved by an (s, t)-walk. Then, $\begin{bmatrix} s \\ t \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix}$ for some nonnegative integers u and v. By Lemma 3.3 we have

$$\begin{bmatrix} u \\ v \end{bmatrix} \ge M^{-1} \begin{bmatrix} r(p_{ki}) \\ r(p_{ki}) \end{bmatrix} = \begin{bmatrix} b(C_2)r(p_{k,i}) - r(C_2)b(p_{k,i}) \\ r(C_1)b(p_{k,i}) - b(C_1)r(p_{k,i}) \end{bmatrix} \tag{6}$$

for any path $p_{k,i}$ from vertex v_k to vertex v_i, $i = 1, 2, \ldots, n$. Therefore from (6) we find that

$$\begin{bmatrix} s \\ t \end{bmatrix} = M \begin{bmatrix} u \\ v \end{bmatrix} \ge M \begin{bmatrix} u_0 \\ v_0 \end{bmatrix}. \tag{7}$$

Equation (7) implies

$$\begin{aligned} \gamma_{D^{(2)}}(v_k) &= s + t \\ &\ge \left(r(C_1) + b(C_1)\right)u_0 + \left(r(C_2) + b(C_2)\right)v_0 \\ &= \ell(C_1)u_0 + \ell(C_2)v_0. \end{aligned}$$

4. Main Results

This section discusses vertex exponents of primitive ministrong 2-digraph $D^{(2)}$ whose underlying digraph is the primitive extremal ministrong digraph in Theorem 2.1. Since $D^{(2)}$ is primitive, by Equation (2) the ministrong 2-digraph $D^{(2)}$ has at most two blue arcs. We split our discussion into two cases, the case when $D^{(2)}$ has one blue arc and the case when $D^{(2)}$ has two blue arcs.

We first consider the case where $D^{(2)}$ has only one blue arc. Notice that when $D^{(2)}$ has only one blue arc, the blue arc must lie on the path $p_{n-3,1}$ of length $n-4$ from vertex v_{n-3} to vertex v_1. By Corollary 3.4, the exponent of a vertex depends heavily on how large the expression $u_0 = b(C_2)r(p_{k,i}) - r(C_2)b(p_{k,i})$ and $v_0 = r(C_1)b(p_{k,j}) - b(C_1)r(p_{k,j})$ could be. We note that u_0 will be large when the path p_{ki} from v_k to v_i contains as many red arcs as possible but as few blue arcs as possible. Similarly v_0 will be large when the path p_{kj} from v_k to v_j contains as many blue arcs as possible but as few red arcs as possible.

Theorem 4.1 *Let D the primitive ministrong digraph on n vertices $v_1, v_2, \ldots, v_n$ with exponent $n^2 - 4n + 6$ and let $D^{(2)}$ be a primitive 2-digraph whose underlying digraph is D. If the blue arc of $D^{(2)}$ is the arc $v_j \to v_{j-1}$, $2 \le j \le n-3$, then*

$$\gamma_{D^{(2)}}(v_k) = \begin{cases} n^2 - 4n + 4 + k - j, & \text{if } 1 \le k \le n-2 \\ n^2 - 4n + 3 + k - j, & \text{if } n-1 \le k \le n. \end{cases}$$

Proof: By equation (2) we have $\ell(C_1) = n - 1$ $\ell(C_2) = n - 2$. We show the lower bound for $\gamma_{D^{(2)}}(v_k)$. For $k = 1, 2, \ldots, n$ we consider paths from v_k to v_j and from v_k to v_{j-1} and then we use Corollary 3.4 to set up the lower bound. We split the proof into three cases.

We consider the case where $1 \le k \le j-1$. There are two paths from v_k to v_i, they are an $(n + k - 2 - j, 0)$-path and an $(n + k - 1 - j, 0)$-path. Using the $(n + k - 2 - j, 0)$-path we have $u_0 = n + k - 2 - j$ and using the $(n + k - 1 - j, 0)$-path we find $u_0 = n + k - 1 - j$. We conclude that $u_0 = n + k - 2 - j$. There are two paths from v_k to v_{j-1}. They are an $(n + k - 2 - j, 1)$-path and an $(n + k - 1 - j, 1)$-path. Using the $(n + k - 2 - j, 1)$-path we have $v_0 = j - k$ and using $(n + k - 1 - j, 1)$-path we find $v_0 = j - k - 1$. We conclude that $v_0 = j - k - 1$. Corollary 3.4 implies that

$$\begin{bmatrix} s \\ t \end{bmatrix} \ge M \begin{bmatrix} n + k - 2 - j \\ j - 1 - k \end{bmatrix} = \begin{bmatrix} n^2 - 5n + 7 + k - j \\ n - 3 \end{bmatrix}. \tag{8}$$

From (8) we have

$$\gamma_{D^{(2)}}(v_k) \ge n^2 - 4n + 4 + k + j \tag{9}$$

for $1 \le k \le j-1$.

We now assume that $j \le k \le n-2$. There is only one path from v_k to v_j, namely the $(k - j, 0)$-path. Using this path we have $u_0 = k - j$. There is only one path from v_k to v_{j-1}, namely the $(k - j, 1)$-path from v_k to v_{j-1}. Using this path we find $v_0 = n - k + j - 2$. By Corollary 3.4

$$\gamma_{D^{(2)}}(v_k) \ge (n-1)(k-j)+(n-2)(n-k+j-2)$$
$$= n^2-4n+4+k-j \qquad (10)$$

for $j \le k \le n-2$.

Finally we assume that $n-1 \le k \le n$. There is only one path from v_k to v_j and there is only one path from v_k to v_{j-1}. Considering the $(k-1-j,\ 0)$-path from v_k to v_j we have $u_0 = k-1-j$. Considering the $(k-1-j,\ 1)$-path from v_k to v_{j-1}, we have $v_0 = n-k+j-1$. By Corollary 3.4 we have

$$\gamma_{D^{(2)}}(v_k) \ge (n-1)(k-1-j)+(n-2)(n-k+j-1)$$
$$= n^2-4n+4+k-j \qquad (11)$$

for $k = n-1, n$.

Hence from (9), (10) and (11) we conclude that

$$\gamma_{D^{(2)}}(v_k) \ge n^2-4n+4+k-j \qquad (12)$$

for all $k = 1, 2, \ldots, n$.

We next show the upper bonds. First we show that $\gamma_{D^{(2)}}(v_1) \le n^2-4n+5-j$ and then we use Proposition 3.2 to get the other bounds. For $i = 1, 2, \ldots, n$ let $p_{1,i}$ be a path from v_1 to v_i. We consider the system of equations

$$Mz + \begin{bmatrix} r(p_{1,i}) \\ b(p_{1,i}) \end{bmatrix} = \begin{bmatrix} n^2-5n+8-j \\ n-3 \end{bmatrix}. \qquad (13)$$

The solution to the system (13) is the integer vector

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} n-1-j-r(p_{1,i})+b(p_{1,i})(n-3) \\ j-2+r(p_{1,i})-b(p_{1,i})(n-2) \end{bmatrix}. \qquad (14)$$

When $b(p_{1,i}) = 0$, there is a path from v_1 to v_i with $r(p_{1,i}) \le n-1-j$. This and equation (14) imply $z_1 \ge 0$. When $b(p_{1,i}) = 1$, then all paths from v_1 to v_i have the property that $r(p_{1,i}) \ge n-j$. This and equation (14) imply $z_2 \ge 0$. Therefore, the system (13) has a nonnegative integer solution. By Proposition 3.1 there is an $(n^2-5n+8-j,\ n-3)$-walk from the vertex v_1 to vertex v_i for all $i = 1, 2, \ldots, n$. Hence $\gamma_{D^{(2)}}(v_1) \le n^2-4n+5-j$. We now conclude that $\gamma_{D^{(2)}}(v_1) = n^2-4n+5-j$.

For $k = 2, \ldots, n-2$, there is a $(k-2,\ 1)$-path of length $k-1$ from v_k to v_i. Proposition 3.2 implies that $\gamma_{D^{(2)}}(v_1) \le n^2-4n+4+k-j$. For $k = n-1, n$, there is a $(k-3,\ 1)$-path of length $k-2$ from v_k to v_1. By Proposition 3.2 $\gamma_{D^{(2)}}(v_k) \le n^2-4n+4+k-j$ for all $k = 2, \ldots, n$.

Hence we conclude that

$$\gamma_{D^{(2)}}(v_k) \le n^2-4n+4+k-j \qquad (15)$$

for all $k = 1, 2, \ldots, n$.

From (12) and (15) we finally conclude that $\gamma_{D^{(2)}}(v_k) = n^2-4n+4+k-j$ for all $k = 1, 2, \ldots, n$.

We now discuss the case where $D^{(2)}$ has two blue arcs. We note that one of the blue arcs must lie on the path $v_1 \to v_n \to v_{n-1} \to v_{n-3}$ and the other must lie on the path $v_1 \to v_{n-2} \to v_{n-3}$. We split the proof into six cases depending on the position of the two blue arcs.

Theorem 4.2 *Let D be the primitive ministrong digraph on n vertices $v_1, v_2, \ldots, v_n$ with exponent n^2-4n+6 and let $D^{(2)}$ be a primitive 2-digraph whose underlying digraph is D. If the blue arcs of $D^{(2)}$ are the arcs $v_1 \to v_{n-2}$ and $v_1 \to v_n$, then*

$$\gamma_{D^{(2)}}(v_k) = \begin{cases} n^2-4n+3+k, & \text{if } 1 \le k \le n-2 \\ n^2-4n+2+k, & \text{if } n-1 \le k \le n. \end{cases}$$

Proof: We first show the lower bound. Let v_k, $1 \le k \le n$ be any vertex in $D^{(2)}$. We use the path from v_k to v_{n-2} and the path from v_k to v_1 in order to get the value of u_0 and v_0 in Corollary 3.4. Notice that for any $k = 1, 2, \ldots, n$, there is a unique path from v_k to v_{n-2} and there is a unique path from v_k to v_1. We split the proof into two cases when $1 \le k \le n-2$ and when $k = n-1, n$.

We first discuss case where $1 \le k \le n-2$. Considering the $(k-1,\ 1)$-path $p_{k,n-2}$ from v_k to v_{n-2} we have $v_0 = n-k-1$. Considering the $(k-1,\ 0)$-path $p_{k,1}$ from v_k to v_1 we have $u_0 = k-1$. Corollary 3.4 implies that

$$\begin{bmatrix} s \\ t \end{bmatrix} \ge \begin{bmatrix} n-2 & n-3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} k-1 \\ n-k-1 \end{bmatrix} = \begin{bmatrix} n^2-5n+5+k \\ n-2 \end{bmatrix}. \qquad (16)$$

From (16) we have

$$\gamma_{D^{(2)}}(v_k) \ge n^2-4n+3+k \qquad (17)$$

for $1 \le k \le n-2$.

We now assume that $k = n-1, n$. Using the $(k-2,\ 1)$-path $p_{k,n-2}$ from v_k to v_{n-2} we find that $v_0 = n-k$. Using the $(k-2,\ 0)$-path $p_{k,1}$ from v_k to v_1 we have $u_0 = k-2$. By Corollary 3.4 we have

$$\gamma_{D^{(2)}}(v_k) \ge (n-1)(k-2)+(n-2)(n-k)$$
$$= n^2-4n+2+k \qquad (18)$$

For $k = n-1, n$.

Hence from (17) and (18) we now have

$$\gamma_{D^{(2)}}(v_k) \ge \begin{cases} n^2-4n+3+k, & \text{if } 1 \le k \le n-2 \\ n^2-4n+2+k, & \text{if } n-1 \le k \le n. \end{cases} \qquad (19)$$

We show the upper bounds. We first show that $\gamma_{D^{(2)}}(v_1) = n^2-4n+4$ and then we use Proposition 3.2 to get the other bounds. For $i = 1, 2, \ldots, n$, let $p_{1,i}$ be a path from v_1 to v_i. The system of equations

$$Mz + \begin{bmatrix} r(p_{1,i}) \\ b(p_{1,i}) \end{bmatrix} = \begin{bmatrix} n^2-5n+6 \\ n-2 \end{bmatrix} \qquad (20)$$

has integer solution

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} b(p_{1,i})(n-3)-r(p_{1,i}) \\ n-2+r(p_{1,i})-b(p_{1,i})(n-2) \end{bmatrix}. \qquad (21)$$

We note that for each $i = 1, 2, \ldots, n$ there is a path $p_{1,i}$ with $b(p_{1,i}) = 1$ and $r(p_{1,i}) \le n-3$. This and equation (21) imply $z_1 \ge 0$ and $z_2 \ge 0$. Hence system (20) has a nonnegative integer solution. Proposition 3.1 implies that $\gamma_{D^{(2)}}(v_1) \le n^2-4n+4$. We can now conclude that $\gamma_{D^{(2)}}(v_1) = n^2-4n+4$.

Since for each $k = 2, 3, \ldots, n-2$, there is a $(k-1,\ 0)$-walk of length $k-1$ from v_k to v_1, then by Proposition 3.2 we have $\gamma_{D^{(2)}}(v_k) \le n^2-4n+3+k$. For each $k = n-1, n$, there is a $(k-2,\ 0)$-path of length $k-2$ from v_k to v_1. Proposition 3.2 implies $\gamma_{D^{(2)}}(v_k) \le n^2-4n+2+k$. Therefore, we have

$$\gamma_{D^{(2)}}(v_k) \le \begin{cases} n^2-4n+3+k, & \text{if } 1 \le k \le n-2 \\ n^2-4n+2+k, & \text{if } n-1 \le k \le n. \end{cases} \qquad (22)$$

From (19) and (22) we now conclude that

$$\gamma_{D^{(2)}}(v_k)=\begin{cases} n^2-4n+3+k, & if\ 1\le k\le n-2 \\ n^2-4n+2+k, & if\ n-1\le k\le n. \end{cases}$$

Theorem 4.3 *Let D be the primitive ministrong digraph on n vertices $v_1, v_2, \ldots, v_n$ with exponent n^2-4n+6 and let $D^{(2)}$ be a primitive 2-digraph whose underlying digraph is D. If the blue arcs of $D^{(2)}$ are the arcs $v_1 \rightarrow v_{n-2}$ and $v_n \rightarrow v_{n-1}$, then*

$$\gamma_{D^{(2)}}(v_k)=\begin{cases} n^2-3n+2+k, & if\ 1\le k\le n-2 \\ n^2-3n+1+k, & if\ n-1\le k\le n. \end{cases}$$

Proof. We show the lower bounds. For $k = 1, 2, \ldots, n,$ we use the path from v_k to v_{n-2} and the path from v_k to v_n to set up the value of u_0 and v_0 in Corollary 3.4. We note that for any $k = 1, 2, \ldots, n$ there is a unique path from v_k to v_n and a unique path from v_k to v_{n-2}.

We first set up the case where $1\le k \le n-2$. Considering the $(k-1, 1)$-path $p_{k,n-2}$ from v_k to v_{n-2} we conclude that $v_0 = n-k-1$. Considering the $(k, 0)$-path $p_{k,n}$ from v_k to v_n we conclude that $u_0 = k$. By Corollary 3.4 we have

$$\begin{bmatrix} s \\ t \end{bmatrix} \ge \begin{bmatrix} n-2 & n-3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} k \\ n-k-1 \end{bmatrix} = \begin{bmatrix} n^2-4n+3+k \\ n-1 \end{bmatrix}. \qquad (23)$$

From (23) we have

$$\gamma_{D^{(2)}}(v_k)\ge n^2-3n+2+k \qquad (24)$$

for $1\le k\le n-2$.

We now assume that $k = n-1$. Using the $(k-2, 1)$-path from v_k to v_{n-2} we have $v_0 = n-k$. Using the $(k-1, 0)$-path from v_k to v_n we find $u_0 = k-1$. Corollary 3.4 implies that

$$\gamma_{D^{(2)}}(v_k)\ge (n-1)(k-1)+(n-2)(n-k)$$
$$= n^2-3n+1+k \qquad (25)$$

for $k = n-1$.

Finally let $k = n$. Using the $(n-3, 2)$-path from v_k to v_{n-2} we find that $v_0 = n-1$. By considering the $(n-2,1)$-path $p_{k,n}$ from v_k to v_n we have $u_0 = 1$. Corollary 3.4 implies that

$$\gamma_{D^{(2)}}(v_k)\ge (n-1)(1)+(n-2)(n-1)$$
$$= n^2-2n+1 = n^2-3n+1+k \qquad (26)$$

for $k = n$.

Hence from (24), (25) and (26) we now conclude that

$$\gamma_{D^{(2)}}(v_k)\ge\begin{cases} n^2-3n+2+k, & if\ 1\le k\le n-2 \\ n^2-3n+1+k, & if\ n-1\le k\le n. \end{cases} \qquad (27)$$

We now discuss the upper bounds. We first show that $\gamma_{D^{(2)}}(v_1)\le n^2-3n+3$ and we use Proposition 3.2 to get the other bounds. For $i = 1, 2, \ldots, n$ let $p_{1,i}$ be a path from v_1 to v_i. The system of equations

$$Mz+\begin{bmatrix} r(p_{1,i}) \\ b(p_{1,i}) \end{bmatrix} = \begin{bmatrix} n^2-4n+4 \\ n-1 \end{bmatrix} \qquad (28)$$

has integer solution

$$z=\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1-r(p_{1,i})+b(p_{1,i})(n-3) \\ n-2+r(p_{1,i})-b(p_{1,i})(n-2) \end{bmatrix}. \qquad (29)$$

For each $i = 1, 2, \ldots, n$ there is a path $p_{1,i}$ from v_1 to v_i with $b(p_{1,i}) \le 1$ and $r(p_{1,i}) \le n-3$. Since $b(p_{1,i}) \le 1$, from (29) we have $z_2 \ge 0$. If for some $i = 1, 2, \ldots, n$ we have $b(p_{1,i}) = 1$, then $r(p_{1,i}) = 1$. This and (29) imply $z_1 \ge 0$. Therefore, the system (28) has a nonnegative integer solution. By Proposition 3.1, we have that $\gamma_{D^{(2)}}(v_1)\le n^2-3n+3$. Hence, we can conclude that $\gamma_{D^{(2)}}(v_1)= n^2-3n+3$.

We note that for each $k = 2, 3, \ldots, n-2$ there is a $(k-1, 0)$-path of length $k-1$ from v_k to v_1. By Proposition 3.2 $\gamma_{D^{(2)}}(v_k)\le n^2-3n+2+k$. For $k = n-1$ the shortest path from v_k to v_1 is a $(k-2, 0)$-path of length $k-2$. Proposition 3.2 implies that $\gamma_{D^{(2)}}(v_k)\le n^2-3n+1+k$. Finally for $k = n$, the shortest path from v_k to v_1 is a $(k-3, 1)$-path of length $k-2$. Proposition 3.2 implies that $\gamma_{D^{(2)}}(v_k)\le n^2-3n+1+k$. Therefore, we now have

$$\gamma_{D^{(2)}}(v_k)\le\begin{cases} n^2-3n+2+k, & if\ 1\le k\le n-2 \\ n^2-3n+1+k, & if\ n-1\le k\le n. \end{cases} \qquad (30)$$

Now from (27) and (30) we conclude that

$$\gamma_{D^{(2)}}(v_k)=\begin{cases} n^2-3n+2+k, & if\ 1\le k\le n-2 \\ n^2-3n+1+k, & if\ n-1\le k\le n. \end{cases}$$

Theorem 4.4 *Let D be the primitive ministrong digraph on n vertices $v_1, v_2, \ldots, v_n$ with exponent n^2-4n+6 and let $D^{(2)}$ be a primitive 2-digraph whose underlying digraph is D. If the blue arcs of $D^{(2)}$ are the arcs $v_1 \rightarrow v_{n-2}$ and $v_{n-1} \rightarrow v_{n-3}$, then*

$$\gamma_{D^{(2)}}(v_k)=\begin{cases} n^2-2n+1+k, & if\ 1\le k\le n-2 \\ n^2-2n+k\ \ \ , & if\ n-1\le k\le n. \end{cases}$$

Proof: We consider the lower bound. For $k = 1, 2, \ldots, n$ we use a path from v_k to v_{n-2} and a path from v_k to v_{n-1} to set up the value of u_0 and v_0 in Corollary 3.4. We note that there is a unique path from v_k to v_{n-2} and there is also a unique path from v_k to v_{n-1}.

First we assume that $1\le k \le n-2$. Using the $(k-1, 1)$-path from v_k to v_{n-2}, we have that $v_0 = n-k-1$. Using the $(k+1,0)$-path from v_k to v_{n-1} we find that $u_0 = k+1$. Therefore By Corollary 3.4 we have

$$\begin{bmatrix} s \\ t \end{bmatrix} \ge M\begin{bmatrix} k+1 \\ n-k-1 \end{bmatrix} = \begin{bmatrix} n^2-3n+1+k \\ n \end{bmatrix}. \qquad (31)$$

From (31) we have

$$\gamma_{D^{(2)}}(v_k)\ge n^2-2n+1+k \qquad (32).$$

for $1\le k\le n-2$.

Now we assume that $k = n-1$. Considering the $(k-3, 2)$-path from v_k to v_{n-2}, we have $v_0 = k+1$. Considering the $(k-1, 1)$-path from v_k to v_{n-1}, we find that $u_0 = 1$. Thus Corollary 3.4 implies that

$$\gamma_{D^{(2)}}(v_k)\ge (n-1)+(n-2)n = n^2-2n+k \qquad (33)$$

for $k = n-1$.

Finally, we assume that $k = n$. Considering the $(1,0)$-path from v_k to v_{n-1}, we have that $u_0 = 1$. Considering the $(n-3, 2)$-path from v_k to u_{n-2}, we have $v_0 = n-1$. By Corollary 3.4,

$$\begin{bmatrix} s \\ t \end{bmatrix} \ge \begin{bmatrix} n-2 & n-3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ n-1 \end{bmatrix} = \begin{bmatrix} n^2-3n+1 \\ n \end{bmatrix}. \qquad (34)$$

From (34) we have $\gamma_{D^{(2)}}(v_k)\ge n^2-2n+1$. For $i = 1, 2, \ldots, n$, let $p_{n,i}$ be a path from v_n to v_i. The solution to the system of equations

$$Mz+\begin{bmatrix} r(p_{1,i}) \\ b(p_{1,i}) \end{bmatrix}=\begin{bmatrix} n^2-3n+1 \\ n \end{bmatrix} \quad (35)$$

is

$$z=\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} 1-r(p_{n,i})+b(p_{n,i})(n-3) \\ n-1+r(p_{n,i})-b(p_{n,i})(n-2) \end{bmatrix}. \quad (36)$$

We note that the path from v_n to v_{n-1} is a (1,0)-path. This implies for path $p_{n,n-1}$ from v_n to v_{n-1} the solution to the system (35) in (36) is $z_1 = 0$ and $z_2 = n$. But this implies there is no $(n^2 - 3n + 1, n)$-walk from the vertex v_n to vertex v_{n-1}. Hence we now conclude that $\gamma_{D^{(2)}}(v_n) > n^2-2n+1$. We note that the shortest walk from v_n to v_{n-1} containing an $(n^2 - 3n + 1, n)$-walk is an $(n^2 - 2n - 1, n + 1)$-walk. Therefore, we conclude that

$$\gamma_{D^{(2)}}(v_k) \geq n^2-n = n^2-2n+k \quad (37)$$

for $k = n$.

Hence from (32), (33) and (37) we now conclude that

$$\gamma_{D^{(2)}}(v_k) \geq \begin{cases} n^2-2n+1+k, & if\ 1 \leq k \leq n-2 \\ n^2-2n+k, & if\ n-1 \leq k \leq n. \end{cases} \quad (38)$$

We now discuss the upper bounds. We first show that $\gamma_{D^{(2)}}(v_1) \leq n^2-2n+2$ and then we use Proposition 3.2 to show the other bounds. For $i = 1, 2, \ldots, n$. Let $p_{1,i}$ be a path from v_1 to v_i . The integer solution to the system

$$Mz+\begin{bmatrix} r(p_{1,i}) \\ b(p_{1,i}) \end{bmatrix}=\begin{bmatrix} n^2-3n+2 \\ n \end{bmatrix} \quad (39)$$

is

$$z=\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} 2-r(p_{1,i})+b(p_{1,i})(n-3) \\ n-2+r(p_{1,i})-b(p_{1,i})(n-2) \end{bmatrix}. \quad (40)$$

Every path $p_{1,i}$ from v_1 to v_i has the property that $r(p_{1,i}) \leq n - 3$ and $b(p_{1,i}) \leq 1$. This and equation (40) imply $z_2 \geq 0$. Moreover, if $b(p_{1,i}) = 0$, then $r(p_{1,i}) = 2$. This and equation (40) imply $z_1 \geq 0$. We conclude that the system (39) has a nonnegative integer solution. Proposition 3.1 implies that $\gamma_{D^{(2)}}(v_1) \leq n^2-2n+2$. Hence we conclude that $\gamma_{D^{(2)}}(v_1) = n^2-2n+2$.

For $1 \leq k \leq n - 2$, there is a $(k - 1,0)$-path from vertex v_k to vertex v_1. Proposition 3.2 guarantees that $\gamma_{D^{(2)}}(v_k) \leq n^2-2n+1+k$. For $k = n - 1$, n, there is a $(k - 3,1)$-path from v_k to v_1. Proposition 3.2 implies that $\gamma_{D^{(2)}}(v_k) \leq n^2-2n+k$. Hence we now have

$$\gamma_{D^{(2)}}(v_k) \leq \begin{cases} n^2-2n+1+k, & if\ 1 \leq k \leq n-2 \\ n^2-2n+k, & if\ n-1 \leq k \leq n. \end{cases} \quad (41)$$

Now from (38) and (41) we conclude that

$$\gamma_{D^{(2)}}(v_k) = \begin{cases} n^2-2n+1+k, & if\ 1 \leq k \leq n-2 \\ n^2-2n+k, & if\ n-1 \leq k \leq n. \end{cases}$$

Theorem 4.5 *Let D be the primitive ministrong digraph on n vertices $v_1, v_2, \ldots, v_n$ with exponent $n^2 - 4n + 6$ and let $D^{(2)}$ be a primitive 2-digraph whose underlying digraph is D. If the blue arcs of $D^{(2)}$ are the arcs $v_{n-2} \to v_{n-3}$ and $v_1 \to v_n$, then*

$$\gamma_{D^{(2)}}(v_k) = \begin{cases} n^2-3n+2+k, & if\ 1 \leq k \leq n-2 \\ n^2-3n+1+k, & if\ n-1 \leq k \leq n. \end{cases}$$

Proof: We show the lower bounds. For $k = 1, 2, \ldots, n$ we use the unique path from v_k to v_{n-2} and the unique path from v_k to v_n to get the value of u_0 and v_0 in Corollary 3.4. We split the proof into three cases.

We first consider the case where $1 \leq k \leq n - 3$. Considering the $(k,0)$-path from v_k to v_{n-2}, we have $u_0 = k$. Considering the $(k - 1,1)$-path from v_k to v_n, we have $v_0 = n - k - 1$. By Corollary 3.4 we conclude that

$$\begin{bmatrix} s \\ t \end{bmatrix} \geq \begin{bmatrix} n-2 & n-3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} k \\ n-k-1 \end{bmatrix}=\begin{bmatrix} n^2-4n+3+k \\ n-1 \end{bmatrix}. \quad (42)$$

From (42) we have

$$\gamma_{D^{(2)}}(v_k) \geq n^2-3n+2+k \quad (43)$$

for $1 \leq k \leq n - 3$.

We consider the case where $k = n - 2$. Using the $(n - 3,1)$-path from vertex v_{n-2} to itself, we have $u_0 = 0$. Using the $(n - 4,2)$-path from vertex v_{n-2} to vertex v_n, we have $v_0 = n$. Corollary 3.4 implies that

$$\gamma_{D^{(2)}}(v_k) \geq (n-2)n = n^2-2n = n^2-3n+2+k \quad (44)$$

for $k = n - 2$.

Finally we consider the case where $k = n - 1$, n. Considering the $(k - 1,0)$-path from v_k to v_{n-2}, we have that $u_0 = k - 1$. Considering the $(k - 2,1)$-path from v_k to v_n, we find that $v_0 = n - k$. Corollary 3.4 implies that

$$\gamma_{D^{(2)}}(v_k) \geq (n-2)(k-1)+(n-3)(n-k)$$
$$= n^2-3n+1+k \quad (45)$$

for $k = n - 1$, n.

Now from (43), (44) and (45) we have that

$$\gamma_{D^{(2)}}(v_k) \geq \begin{cases} n^2-3n+2+k, & if\ 1 \leq k \leq n-2 \\ n^2-3n+1+k, & if\ n-1 \leq k \leq n. \end{cases} \quad (46)$$

We now discuss the upper bound. We first show that $\gamma_{D^{(2)}}(v_1) = n^2-3n+3$ and use Proposition 3.2 to get the other bounds. For $i = 1, 2, \ldots, n$. let $p_{1,i}$ be a path from v_1 to v_i. The solution of the system

$$Mz+\begin{bmatrix} r(p_{1,i}) \\ b(p_{1,i}) \end{bmatrix}=\begin{bmatrix} n^2-4n+4 \\ n-1 \end{bmatrix} \quad (47)$$

is the integer vector

$$z=\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} 1-r(p_{1,i})+b(p_{1,i})(n-3) \\ n-2+r(p_{1,i})-b(p_{1,i})(n-2) \end{bmatrix}. \quad (48)$$

For each $i = 1, 2, \ldots, n$, there is a path $p_{1,i}$ with the property that $b(p_{1,i}) \leq 1$ and $r(p_{1,i}) \leq n - 3$. Moreover, when $b(p_{1,i}) = 0$, then $r(p_{1,i}) = 1$. This and equation (48) imply $z_1 \geq 0$. We also note that when $b(p_{1,i}) = 1$, then $r(p_{1,i}) \leq n - 3$. This and equation (48) imply $z_2 \geq 0$. Therefore, the system (47) has a nonnegative integer solution. By Proposition 3.1 $\gamma_{D^{(2)}}(v_1) \leq n^2-3n+3$ and hence we conclude that $\gamma_{D^{(2)}}(v_1) = n^2-3n+3$.

For each $k= 2, 3, .., n - 3$, there is a $(k - 1,0)$-path of length $k - 1$ from v_k to v_1. Proposition 3.2 implies that $\gamma_{D^{(2)}}(v_k) \leq n^2-3n+2+k$. For $k = n - 2$, there is a $(k - 2,1)$-path of length $k - 1$ from v_k to v_1. By Proposition 3.2 $\gamma_{D^{(2)}}(v_k) \leq n^2-3n+2+k$. Finally, when $k = n - 1$, n there is a $(k - 2,0)$-path of length $k-2$ from vertex v_k to vertex v_1. By Proposition 3.2 $\gamma_{D^{(2)}}(v_k) \leq n^2-3n+1+k$; therefore, we now have

$$\gamma_{D^{(2)}}(v_k) \le \begin{cases} n^2-3n+2+k, & \text{if } 1\le k\le n-2 \\ n^2-3n+1+k, & \text{if } n-1\le k\le n. \end{cases} \quad (49)$$

From (46) and (49) we now conclude that

$$\gamma_{D^{(2)}}(v_k) = \begin{cases} n^2-3n+2+k, & \text{if } 1\le k\le n-2 \\ n^2-3n+1+k, & \text{if } n-1\le k\le n. \end{cases}$$

Theorem 4.6 *Let D be the primitive ministrong digraph on n vertices $v_1, v_2, \ldots, v_n$ with exponent $n^2 - 4n + 6$ and let $D^{(2)}$ be a primitive 2-digraph whose underlying digraph is D. If the blue arcs of $D^{(2)}$ are the arcs $v_{n-2} \to v_{n-3}$ and $v_n \to v_{n-1}$, then*

$$\gamma_{D^{(2)}}(v_k) = \begin{cases} n^2-4n+4+k, & \text{if } 1\le k\le n-2 \\ n^2-4n+3+k, & \text{if } n-1\le k\le n. \end{cases}$$

Proof: We show the lower bounds. For k = 1, 2, …, n. We use the unique path from v_k to v_{n-2} and the unique path from v_k to v_{n-1} to set the value of u_0 and v_0 in Corollary 3.4. We split the proof into four cases.

We first let $1 \le k \le n - 3$. Considering the $(k,0)$-path from v_k to v_{n-2}, we have $u_0 = k$. Considering the $(k,1)$-path from v_k to v_{n-1}, we find $v_0 = n - k - 2$. Corollary 3.4 implies that

$$\begin{bmatrix} s \\ t \end{bmatrix} \ge \begin{bmatrix} n-2 & n-3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} k \\ n-k-2 \end{bmatrix} = \begin{bmatrix} n^2-5n+6+k \\ n-2 \end{bmatrix}. \quad (50)$$

From (50) we have

$$\gamma_{D^{(2)}}(v_k) \ge n^2-4n+4+k \quad (51)$$

for $1 \le k \le n-3$.

We consider the case where $k = n - 2$. Using the $(k - 1,1)$-path from v_k to v_{n-2}, we have that $u_0 = k-n+2$. Using the $(k - 1,2)$-path from v_k to v_{n-1}, we find $v_0 = 2n-k-3$. By Corollary 3.4 we conclude that

$$\gamma_{D^{(2)}}(v_k) = (n-1)(k-n+2)+(n-1)(2n-k-1) = n^2-4n+4+k \quad (52)$$

for $k = n - 2$.

We now let $k = n - 1$. Considering the $(k - 1,0)$-path from v_k to v_n, we have $u_0 = k - 1$. Considering the $(k - 1,1)$-path from v_k to v_{n-1}, we have $v_0 = n - k - 1$. By Corollary 3.4

$$\gamma_{D^{(2)}}(v_k) \ge (n-1)(k-1)+(n-2)(n-k-1) = n^2-4n+3+k \quad (53)$$

for $k = n - 1$.

Finally let $k = n$. Considering the $(0,1)$-path from v_k to v_{n-1}, we have $v_0 = n - 2$. Considering the $(k - 2,1)$-path from v_k to v_{n-2}, we find that $u_0 = 1$. By Corollary 3.4

$$\gamma_{D^{(2)}}(v_k) \ge (n-1)(1)+(n-2)(n-2) = n^2-3n+3 = n^2-4n+3+k \quad (54)$$

From (51), (52), (53) and (54) we now can conclude that

$$\gamma_{D^{(2)}}(v_k) \ge \begin{cases} n^2-4n+4+k, & \text{if } 1\le k\le n-2 \\ n^2-4n+3+k, & \text{if } n-1\le k\le n. \end{cases} \quad (55)$$

We now set up the upper bound. We first show that $\gamma_{D^{(2)}}(v_1) \le n^2-4n+5$ and then use Proposition 3.2 to set up the other bounds. For i = 1, 2, …, n let $p_{1,i}$ be a path from v_1 to v_i. The system of equation

$$Mz + \begin{bmatrix} r(p_{1,i}) \\ b(p_{1,i}) \end{bmatrix} = \begin{bmatrix} n^2-5n+7 \\ n-2 \end{bmatrix} \quad (56)$$

has integer solution

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1-r(p_{1,i})+b(p_{1,i})(n-3) \\ n-3+r(p_{1,i})-b(p_{1,i})(n-2) \end{bmatrix}. \quad (57)$$

We note that if $b(p_{1,i}) = 0$, then $r(p_{1,i}) = 1$. This and equation (57) imply $z_1 \ge 0$. Moreover if $b(p_{1,i}) = 1$, then $r(p_{1,i}) \ge 1$. This and equation (57) imply $z_2 \ge 0$. Therefore the system (56) has a nonnegative integer solution. Hence Proposition 3.1 guarantees that $\gamma_{D^{(2)}}(v_1) \le n^2-4n+5$. Therefore, we now conclude that $\gamma_{D^{(2)}}(v_1) = n^2-4n+5$.

Note that for k = 2, …, n – 3, there is a $(k - 1, 0)$-path of length $k - 1$ from v_k to v_1. Proposition 3.2 implies that $\gamma_{D^{(2)}}(v_2) \le n^2-4n+4+k$. For $k = n - 2$, there is a $(k - 2,1)$-path from v_k to v_1 of length $k - 1$. By Proposition 3.2 $\gamma_{D^{(2)}}(v_k) \le n^2-4n+4+k$. For $k = n - 1$, there is a $(k - 2,0)$-path of length $k - 2$ from the vertex v_k to the vertex v_1. By Proposition 3.2 $\gamma_{D^{(2)}}(v_k) \le n^2-4n+3+k$. Finally for $k = n$, there is a $(k - 3,1)$-path of length $k - 2$ from v_k to v_1. By Proposition 3.2 we now have that $\gamma_{D^{(2)}}(v_k) \le n^2-4n+3+k$. Therefore, we know have

$$\gamma_{D^{(2)}}(v_k) \le \begin{cases} n^2-4n+4+k, & \text{if } 1\le k\le n-2 \\ n^2-4n+3+k, & \text{if } n-1\le k\le n. \end{cases} \quad (58)$$

From (55) and (58) we conclude that

$$\gamma_{D^{(2)}}(v_k) = \begin{cases} n^2-4n+4+k, & \text{if } 1\le k\le n-2 \\ n^2-4n+3+k, & \text{if } n-1\le k\le n. \end{cases}$$

Theorem 4.7 *Let D be the primitive ministrong digraph on n vertices $v_1, v_2, \ldots, v_n$ with exponent n^2 - $4n + 6$ and let $D^{(2)}$ be a primitive 2-digraph whose underlying digraph is D. If the blue arcs of $D^{(2)}$ are the arcs $v_{n-2} \to v_{n-3}$ and $v_{n-1} \to v_{n-3}$, then*

$$\gamma_{D^{(2)}}(v_k) = \begin{cases} n^2-4n+5+k, & \text{if } 1\le k\le n-2 \\ n^2-4n+4+k, & \text{if } n-1\le k\le n. \end{cases}$$

Proof: We set up the lower bounds. We use a path from v_k to v_{n-1} and a path from v_k to v_{n-3} to get the value of u_0 and v_0 in Corollary 3.4. We split the proof into four cases.

First we consider the case where $1 \le k \le n - 3$. Notice that there are two paths from v_k to v_{n-3} one is a $(k,1)$-path and the other is a $(k + 1,1)$-path. Using the $(k + 1,0)$-path from v_k to v_{n-1}, we have $u_0 = k + 1$. Using the $(k,1)$-path from v_k to v_{n-3}, we have $v_0 = n - k - 2$. Using the $(k + 1,1)$-path from v_k to v_{n-3} we find that $v_0 = n - k - 3$. So we conclude that $v_0 = n - k - 3$ and by Corollary 3.4 we have

$$\begin{bmatrix} s \\ t \end{bmatrix} \ge M\begin{bmatrix} k+1 \\ n-k-3 \end{bmatrix} = \begin{bmatrix} n^2-5n+7+k \\ n-2 \end{bmatrix}. \quad (59)$$

From (59) we have

$$\gamma_{D^{(2)}}(v_k) \ge n^2-4n+5+k \quad (60)$$

for $1 \le k \le n - 3$.

We now assume that $k = n - 2$. Considering the $(k,1)$-path from v_k to v_{n-1}, we have $u_0 = 1$. Considering the $(0,1)$-path from v_k to v_{n-3}, we have $v_0 = n - 2$. Hence Corollary 3.4 implies that

$$\gamma_{D^{(2)}}(v_k) \ge (n-1)(1)+(n-2)(n-2) = n^2-3n+3 = n^2-4n+5+k \quad (61)$$

for $k = n - 2$.

We consider the case where $k = n - 1$. Using the $(k - 1,1)$-path from v_k to v_{n-1}, we have $u_0 = 1$. Using the (0,1)-path from v_k to v_{n-3}, we have $v_0 = n - 2$, By Corollary 3.4 we conclude

$$\gamma_{D^{(2)}}(v_k) \geq (n-1)+(n-2)(n-2)$$
$$= n^2 - 3n + 3 = n^2 - 4n + 4 + k \qquad (62)$$

for $k = n - 1$.

Finally let $k = n$. Considering the (1,0)-path from v_k to v_{n-1}, we have $u_0 = 1$. Considering the (1,1)-path from v_k to v_{n-3}, we have $v_0 = n - 3$. Corollary 3.4 implies that

$$\begin{bmatrix} s \\ t \end{bmatrix} \geq \begin{bmatrix} n-2 & n-3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ n-3 \end{bmatrix} = \begin{bmatrix} n^2-5n+7 \\ n-2 \end{bmatrix}. \qquad (63)$$

From (63) $\gamma_{D^{(2)}}(v_k) \geq n^2 - 4n + 5 = n^2 - 5n + 5 + k$. Notice that the system of equations

$$Mz + \begin{bmatrix} r(p_{k,i}) \\ b(p_{k,i}) \end{bmatrix} = \begin{bmatrix} n^2-5n+7 \\ n-2 \end{bmatrix} \qquad (64)$$

has integer solution

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 - r(p_{k,i}) + b(p_{k,i})(n-3) \\ n-3+r(p_{k,i}) - b(p_{k,i})(n-2) \end{bmatrix}. \qquad (65)$$

We note that the path from v_n to v_{n-1} is a (1,0)-path. This implies for path $p_{n,n-1}$, the solution to the system (64) in equation (65) is $z_1 = 0$ and $z_2 = n - 2$. But this implies there is no $(n^2 - 5n + 7,\ n - 2)$-walk from the vertex v_n to vertex v_{n-1}, hence $\gamma_{D^{(2)}}(v_k) > n^2 - 4n + 5$. Notice that the shortest walk from v_n to v_{n-1} with at least $n^2 - 5n + 7$ red arcs and at least $n - 2$ blue arcs is an $(n^2 - 4n + 5,\ n - 1)$-walk. Hence we conclude that

$$\gamma_{D^{(2)}}(v_k) \geq n^2 - 3n + 4 = n^2 - 4n + 4 + k \qquad (66)$$

for $k = n$.

Hence from (60), (61), (62) and (66) we now have the lower bound

$$\gamma_{D^{(2)}}(v_k) \geq \begin{cases} n^2-4n+5+k, & \text{if } 1 \leq k \leq n-2 \\ n^2-4n+4+k, & \text{if } n-1 \leq k \leq n. \end{cases} \qquad (67)$$

We next consider the upper bound. We first show that $\gamma_{D^{(2)}}(v_1) = n^2 - 4n + 6$ and then we use Proposition 3.2 to get the other bounds. For $i = 1, 2, \ldots, n$ let $p_{1,i}$ be a path from v_1 to v_i. The system

$$Mz + \begin{bmatrix} r(p_{1,i}) \\ b(p_{1,i}) \end{bmatrix} = \begin{bmatrix} n^2-5n+8 \\ n-2 \end{bmatrix} \qquad (68)$$

has integer solution

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2 - r(p_{1,i}) + b(p_{1,i})(n-3) \\ n-4+r(p_{1,i}) - b(p_{1,i})(n-2) \end{bmatrix}. \qquad (69)$$

Notice that when $b(p_{1,i}) = 0$, then $r(p_{1,i}) = 2$. This and equation (69) imply $z_1 \geq 0$. When $b(p_{1,i}) = 1$, then there is a path $p_{1,i}$ such that $r(p_{1,i}) \geq 2$. This and equation (69) imply $-z_2 \geq 0$. Thus the system (68) has a nonnegative integer solution. By Proposition 3.1 $\gamma_{D^{(2)}}(v_1) \leq n^2 - 4n + 6$. We now conclude that $\gamma_{D^{(2)}}(v_1) = n^2 - 4n + 6$.

For $k = 2, 3, \ldots, n - 3$, there is a $(k - 1,0)$-path of length $k - 1$ from v_k to v_1. Proposition 3.2 implies that $\gamma_{D^{(2)}}(v_k) \leq n^2 - 4n + 5 + k$. When $k = n - 2$, there is a $(k - 2,1)$-path of length $k - 1$ from the vertex v_k to v_1. Proposition 3.2 implies $\gamma_{D^{(2)}}(v_k) \leq n^2 - 4n + 5 + k$. When $k = n - 1$, n, there is a $(k - 3,1)$-path of length $k - 2$ from v_k to v_1. Proposition 3.2 guarantees that $\gamma_{D^{(2)}}(v_k) \leq n^2 - 4n + 4 + k$; Therefore, we now have the upper bound

$$\gamma_{D^{(2)}}(v_k) \leq \begin{cases} n^2-4n+5+k, & \text{if } 1 \leq k \leq n-2 \\ n^2-4n+4+k, & \text{if } n-1 \leq k \leq n. \end{cases} \qquad (70)$$

From (67) and (70) we conclude that

$$\gamma_{D^{(2)}}(v_k) = \begin{cases} n^2-4n+5+k, & \text{if } 1 \leq k \leq n-2 \\ n^2-4n+4+k, & \text{if } n-1 \leq k \leq n. \end{cases}$$

5. Conclusion and Future Research Directions

In this paper we study the vertex exponents of two-colored extremal ministrong digraphs. By comparing the composition of closed and open walks in the two-colored digraph, we get a way in setting up lower and upper bounds for vertex exponents especially for two-colored digraphs consisting of two cycles. Using these bounds Theorem 4.1 shows that if $D^{(2)}$ has only one blue arc, then each of its vertices has exponent that lies on the interval $[n^2 - 5n + 8,\ n^2 - 3n + 1]$. The sequence of Theorem 4.2 to Theorem 4.7 show that if $D^{(2)}$ has two blue arcs, then each of its vertices has exponent lies on the interval $[n^2 - 4n + 4,\ n^2 - n]$.

We note that Gao and Shao [4] have discussed vertex exponents for a class of two-colored digraph whose underlying digraph is the Wielandt digraph. In this paper we discuss vertex exponents for a class of two-colored ministrong digraph on n vertices whose underlying digraph is extremal ministrong digraph with exponent $n^2 - 4n + 6$. There are a lot of open problems on vertex exponents of two-colored digraphs. For example the vertex exponents for classes extremal two-colored ministrong digraphs $D^{(2)}$ on n vertices, with exponent $\exp(D^{(2)}) = (n^3 - 2n^2 + 1)/2$ when n is odd and exponent $(n^3 - 5n^2 + 7n - 2)/2$ when n is even (see Theorem 5 of [6]), have not yet been determined. Shao and Gao [6,16] and Huang and Liu [8] discuss extensively the exponents of classes of two-colored digraphs consisting of two cycles. Similar investigation can be done for the vertex exponents. Finally the vertex exponents of class of two-colored symmetric digraph has not been determined while the exponents of class of two-colored symmetric digraphs has been determined in [17,18].

References

[1] R. A., Brualdi and B. Liu, "Generalized exponents of primitive directed graphs", J. *Graph* Theory 14(4) (1990), 483-499.

[2] R. A. Brualdi and J. A. Ross, "On the exponent of a primitive nearly reducible matrix", *Math. Oper. Res.* **5(2)** (1980), 229-241.

[3] R.A. Brualdi and H.J. Ryser, "Combinatorial matrix theory", Cambridge University Press: Cambridge (1991).

[4] R. A. Brualdi and J. Shao, "Generalized exponents of primitive symmetric digraphs", Discrete Applied Mathematics 74 (1997), 275-293.

[5] E. Fornasini and M. E. Valcher, "Primitivity positive matrix pairs : algebraic characterization graph theoretic description and 2D systems interpretations", *SIAM J. Matrix Anal. Appl.* **19** (1998) 71-88.

[6] Y. Gao and Y. Shao, "Exponents of two-colored digraphs with two cycles", Linear Algebra Appl. 407 (2005), 263-276.

[7] Y. Gao and Y. Shao, "Generalized exponents of primitive two-colored digraphs", *Linear Algebra Appl.* 430 (2009), 1550-1565.

[8] F. Huang and B. Liu, "Exponents of a class of two-colored digraphs with two cycles", Linear Algebra Appl. 429 (2008), 658-672

[9] S. G. Lee, and J. A. Yang, "Bound for 2-exponents of primitive extremal ministrong digraphs", *Commun. Korean. Math. Soc.* **20** (1) (2005) 51-62.

[10] B. Liu, "Generalized exponents of primitive simple graphs", Acta Math. Appl. Sinica 9 (1993) 36-43.

[11] B. Liu, "Generalized exponents of primitive, nearly reducible matrices", Ars Combin., 51 (1999), 229-239

[12] B. Liu and Q. L. Li, "On a conjecture about the generalized exponent of primitive matrices", J. Graph Theory, 18 (1994), 177-179

[13] S. Neufeld and J. Shen, "Some results on generalized exponents", J. Graph Theory, 28 (1998), 215-225

[14] B. L. Shader and S. Suwilo, "Exponents of nonnegative matrix pairs", *Linear Algebra Appl.* 363 (2003), 275-293.

[15] J. Y. Shao and X. J. Wu, "Finiteness conditions for generalized exponents of digraphs", Chinese Ann. Math. Ser. B. 19 (1998), 453-464.

[16] Y. Shao, Y. Gao and L. Sun, "Exponents of a class of two-colored digraphs", *Linear and Multilinear Algebra* 53 (3) (2005), 175-188.

[17] Y. Shao and Y. Gao, "Exponents of symmetric loopless two-colored digraphs", *Linear and Multilinear Algebra* 57 (1) (2009), 65-74.

[18] S. Suwilo, "Notes on exponents of asymmetric two-colored digraphs", J. Combin. Math. Combin. Comput. 71 (2009), 243-256.

[19] S. Suwilo, "Vertex exponents of two-colored extremal ministrong digraphs", Proceedings of The fourth Global Conference on Power Control and Optimization, Kuching December 2010, 82-89

[20] X.J. Wu, J.Y. Shao, Z.M. Yang and X. Z. Zhou, "On the exponents of primitive ministrong digraphs with shortest elementary circuit length *s*", Linear Algebra Appl. 222 (1995), 41-61

[21] S.J. Yang and G.P. Baker, "On the exponents of primitive minimally strong digraph", Linear Algebra Appl. 99 (1988), 177-198

[22] B. Zhou, "Matrices with maximum upper multiexponents in the class of primitive, nearly reducible matrices", Taiwanese J. Math., 2 (1998), 181-190

[23] B. Zhou, "Upper generalized exponents of ministrong digraphs", Taiwanese J. Math. Vol 10 No. 4 (2006), 891-904.